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2007 J. Phys. A: Math. Theor. 40 F935

(http://iopscience.iop.org/1751-8121/40/43/F02)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) F935–F941 doi:10.1088/1751-8113/40/43/F02

FAST TRACK COMMUNICATION

# A criterion for entanglement in two two-level systems

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Received 29 August 2007, in final form 21 September 2007
Published 9 October 2007
Online at stacks.iop.org/JPhysA/40/F935

**Abstract**
The exact entanglement evolution of two two-level systems, coupled to $N$ surrounding spin $\frac{1}{2}$, is analysed with the help of a new criterion relying on the ability of measuring a few simple observables only.

PACS numbers: 03.65.Ud, 03.67.Mn, 75.10.Jm

(Some figures in this article are in colour only in the electronic version)

It is today experimentally possible to realize entangled states in different contexts such as CQED [1–4], metal–superconductor junctions [5, 6] and other solid state systems [7, 8]. A debated and open issue concerns the individuation of efficient methods to detect and quantify entanglement. Since to determine the density operator by tomographic techniques may require considerable efforts, it is desirable to probe and to quantify the occurrence of entanglement with other approaches. To this end, Bell inequalities and witness operators [9–12] could be used as well as criteria relying on the measurement of few observables of clear physical meaning. Methods based on violation of local uncertainty relations [13] or inequalities for variances of observables [14] have been quite recently proposed.

In this communication, we present a criterion for entanglement in two two-level systems coupled to $N$ spin $\frac{1}{2}$ surrounding them. Our results do provide an incisive tool to infer the existence of non-classical correlations from few and simple measurements.

Consider a bipartite system composed of two two-level systems is described by the Pauli operators $\vec{\sigma}_i \equiv \left(\sigma_{\pm}^{(i)}, \sigma_z^{(i)}\right), i = 1, 2$, respectively. Suppose that its density operator, in the factorized basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ of the eigenstates of $\sigma_z^{(1)}\sigma_z^{(2)}$, has the following structure:

$$\rho = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c^* & d & 0 \\ 0 & 0 & 0 & e \end{pmatrix}. \tag{1}$$

The quite simple form of $\rho$, given by equation (1), naturally arises in many physical scenarios not necessarily involving spin $\frac{1}{2}$ systems [2, 3, 5, 15]. Since the concurrence function [16] associated with equation (1) is

$$\mathrm{Conc} = \max[0, 2(|c| - \sqrt{ae})], \tag{2}$$

the existence of entanglement in our system requires that the geometric mean of the two populations $a$ and $e$ becomes less than the coherence amplitude $|c|$. The two probabilities $a$ and $e$ of finding the system in the states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, respectively, determine the mean value of $S_z \equiv \frac{1}{2}\left(\sigma_z^{(1)} + \sigma_z^{(2)}\right)$ and $S_z^2$ as follows:

$$\langle S_z\rangle \equiv \mathrm{tr}\{\rho S_z\} = a - e, \tag{3}$$

$$\langle (S_z)^2\rangle \equiv \mathrm{tr}\{\rho (S_z)^2\} = a + e, \tag{4}$$

which in turn imply that

$$(\Delta S_z)^2 \equiv \langle (S_z)^2\rangle - (\langle S_z\rangle)^2 = (a+e) - (a-e)^2. \tag{5}$$

In accordance with the Landau's condition [17],

$$|c| \leqslant \sqrt{bd}, \tag{6}$$

we may thus state that the presence of entanglement in two two-level systems described by equation (1) necessarily requires

$$\sqrt{ae} < |c| \leqslant \sqrt{bd}. \tag{7}$$

Exploiting such an inequality in equation (5) yields the following upper limit to the variance of $S_z$:

$$(\Delta S_z)^2 < 4bd - (b+d)[(b+d) - 1]. \tag{8}$$

It is of relevance the fact that when the density matrix (1) assumes the special form

$$\rho(t) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & b & 0 \\ 0 & b & b & 0 \\ 0 & 0 & 0 & e \end{pmatrix} \tag{9}$$

the necessary condition (8) simply reduces to

$$(\Delta S_z)^2 < 2b \tag{10}$$

or, equivalently,

$$(\langle S_z\rangle)^2 > 1 - 4b. \tag{11}$$

Since when condition (11) is fulfilled then the concurrence is different from zero, we may claim that equation (11) is a necessary and sufficient condition in order to have entanglement in two two-level systems described by $\rho$ as given by equation (9). Stated another way, the occurrence of entanglement in the system may be checked simply by comparing the square of the mean value of $S_z$ with the population $b$. It is however important to stress that the result we have obtained allows us to immediately deduce that if $b > \frac{1}{4}$, then we can be sure that the two two-level systems are entangled. On the contrary, when $b \leqslant \frac{1}{4}$, we need also to know the mean value of $S_z$. In this case, indeed the bipartite system is entangled if, and only if,

$$-|2b - 1| \leqslant \langle S_z\rangle < -\sqrt{1 - 4b} \tag{12}$$

or, alternatively,

$$\sqrt{1 - 4b} < \langle S_z\rangle \leqslant |2b - 1|. \tag{13}$$

In other words, when $b \leqslant \frac{1}{4}$, the presence of entanglement in the system is compatible only with $S_z$ mean values 'squashed' toward the extremes of its interval of variability $[-1, 1]$. In turn, it implies that under the condition $0 < b \leqslant \frac{1}{4}$, in order to have entanglement, the probability of finding the system in the state $|\uparrow\uparrow\rangle$ must be appreciably different from the probability to find the system in the state $|\downarrow\downarrow\rangle$.

Our criterion may be successfully exploited in order to analyse the entanglement evolution in a spin system describing a physical scenario of interest in many physical contexts. Consider indeed a system constituted of two uncoupled spins $\frac{1}{2}$, denoted by $A$ and $B$ and hereafter called central system, each one interacting with $M-2$ mutually uncoupled spins $\frac{1}{2}$. In particular, we suppose that the two central spins interact with each of the $M-2$ spins in the same way. This system, known as spin star-like system [18, 19], can be described by adopting the following Hamiltonian model:

$$H = H_0 + H_I, \tag{14}$$

with

$$H_0 = \omega(S_z + J_z), \tag{15}$$

$$H_I = \alpha(S_+J_- + S_-J_+), \tag{16}$$

where $S_z$ and $S_\pm \equiv \frac{1}{2}\big(\sigma_\pm^{(1)} + \sigma_\pm^{(2)}\big)$ are spin operators acting in the Hilbert space of the central system and $J_z$ and $J_\pm$ are the collective spin operators describing the other $M-2$ spins. This Hamiltonian model can be successfully used to describe for example electronic spins in semiconductor quantum dot coupled by hyperfine interaction with nuclear spins, or electronic spins bound to phosphorus atoms in a matrix of silica or germanio in the presence of defects [20]. The Hamiltonian model (14) possesses permutational symmetries successfully exploitable to exactly solve the relative time-dependent Schrödinger equation [21, 22]. Suppose that at $t = 0$ the central system is in a common eigenstate of $S^2 = (\vec{S}_1 + \vec{S}_2)^2$ and $S_z$ denoted by $|S, M_S\rangle$. At the same time, the remaining $M-2$ spins are supposed in the state $|J, M_J, \nu\rangle$, eigenstate of the collective angular momentum operators $J^2$ and $J_z$. The index $\nu$, depending on $J$, allows us to distinguish between different states of the coupled angular momentum basis characterized by the same $J$ and $M_J$. The Hamiltonian (14) is invariant by permutation of the two central spins as well as of an arbitrary couple of spins among the $M-2$ of the second block. Moreover, $[S^2, H] = [J^2, H] = [S_z + J_z, H] = \big[J_{\text{int}}^2, H\big] = 0$, $\vec{J}_{\text{int}}$ being an intermediate angular momentum resulting from the coupling of selected at will the individual angular momentum of the $M-2$ spins. At a generic time instant $t$, we can write

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}H_0 t}\,\mathrm{e}^{-\mathrm{i}H_I t}|S, M_S\rangle|J, M_J, \nu\rangle, \tag{17}$$

being $[H_0, H_I] = 0$. The case $S = 0$, and consequently $M_S = 0$, is a trivial one corresponding to an eigenstate of the Hamiltonian whatever the value of $J$ and $M_J$ are. When instead $S = 1$, we have

$$\begin{aligned} H_I^{2n}|1, M_S\rangle|J, M_J, \nu\rangle = {}& \big[\alpha^{2n}(\sqrt{2})^{2n} p_{M_S}\big(p_{M_S}^2 + r_{M_S}^2\big)^{n+\frac{(-1)^{M_S}-1}{2}}\big](p_{M_S}|1, M_S\rangle|J, M_J, \nu\rangle \\ & + r_{M_S}|1, -M_S\rangle|J, M_J + 2M_S, \nu\rangle) \end{aligned} \tag{18}$$

and

$$\begin{aligned} & H_I^{2n+1}|1, M_S\rangle|J, M_J, \nu\rangle \\ & = \begin{cases} \alpha^{2n+1}(\sqrt{2})^{2n+1} p_{M_S}\big(p_{M_S}^2 + r_{M_S}^2\big)^n|1, 0\rangle|J, M_J + M_S, \nu\rangle, & \text{if} \quad M_S \neq 0, \\ \alpha^{2n+1}(\sqrt{2})^{2n+1}\big[p_{-1}\big(p_{-1}^2 + p_1^2\big)^n|1, 1\rangle|J, M_J - 1, \nu\rangle & \\ \quad + p_1\big(p_{-1}^2 + p_1^2\big)^n|1, -1\rangle|J, M_J + 1, \nu\rangle\big], & \text{if} \quad M_S = 0, \end{cases} \end{aligned} \tag{19}$$

with

$$p_s = \sqrt{J(J+1) - M_J(M_J+s)}, \qquad s = \pm 1 \quad p_0 = 1, \tag{20}$$

$$r_s = \sqrt{J(J+1) - (M_J+s)(M_J+2s)}, \qquad s = \pm 1 \quad r_0 = 0. \tag{21}$$

After straightforward calculations, we thus get

$$\begin{aligned}|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\omega(M_S+M_J)t}\bigg\{ & A^{M_S}_{JM_J}(t)|1, M_S\rangle|J, M_J, \nu\rangle + B^{M_S}_{JM_J}(t)|1, -M_S\rangle|J, M_J+2M_S, \nu\rangle \\ & - \mathrm{i}\left(\frac{1-(-1)^{M_S}}{2}\right) C^{M_S}_{JM_J}(t)|1, 0\rangle|J, M_J+M_S, \nu\rangle \\ & - \mathrm{i}\delta_{M_S0}\big[D^{M_S}_{JM_J}(t)|1, 1\rangle|J, M_J-1, \nu\rangle + E^{M_S}_{JM_J}(t)|1, -1\rangle|J, M_J+1, \nu\rangle\big]\bigg\}.\end{aligned} \tag{22}$$

with

$$A^{M_S}_{JM_J}(t) = \left[\frac{p^2_{M_S}}{p^2_{M_S}+r^2_{M_S}}\cos\left(\sqrt{2\left(p^2_{M_S}+r^2_{M_S}\right)}\alpha t\right) + \frac{r^2_{M_S}}{p^2_{M_S}+r^2_{M_S}}\right], \tag{23}$$

$$B^{M_S}_{JM_J}(t) = \frac{p_{M_S} r_{M_S}}{p^2_{M_S}+r^2_{M_S}}\left(\cos\left(\sqrt{2\left(p^2_{M_S}+r^2_{M_S}\right)}\alpha t\right) - 1\right), \tag{24}$$

$$C^{M_S}_{JM_J}(t) = \frac{p_{M_S}}{\sqrt{p^2_{M_S}+r^2_{M_S}}}\sin\left(\sqrt{2\left(p^2_{M_S}+r^2_{M_S}\right)}\alpha t\right), \tag{25}$$

$$D^{M_S}_{JM_J}(t) = \frac{p_{-1}}{\sqrt{p^2_{-1}+p^2_1}}\sin\left(\sqrt{2\left(p^2_{-1}+p^2_1\right)}\alpha t\right), \tag{26}$$

$$E^{M_S}_{JM_J}(t) = \frac{p_1}{\sqrt{p^2_{-1}+p^2_1}}\sin\left(\sqrt{2\left(p^2_{-1}+p^2_1\right)}\alpha t\right). \tag{27}$$

We are interested in the entanglement dynamics of the two central spins $A$ and $B$. Tracing $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ over all the degrees of freedom of the $M-2$ spins around $A$ and $B$, we get the reduced density matrix $\rho_c(t)$ of the central system. It assumes the block diagonal structure given by equation (9), every matrix element being an explicit function of the probability amplitudes (23)–(27). In order to understand this point let us explicitly consider, as an example, the case $M_S = 1$ in correspondence of which equation (22) becomes

$$\begin{aligned}|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\omega(1+M_J)t}\big\{ & A^1_{JM_J}(t)|1, 1\rangle|J, M_J, \nu\rangle \\ & + B^1_{JM_J}(t)|1, -1\rangle|J, M_J+2, \nu\rangle - \mathrm{i}C^1_{JM_J}(t)|1, 0\rangle|J, M_J+1, \nu\rangle\big\}.\end{aligned} \tag{28}$$

The corresponding density matrix $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ can be easily reduced to $\rho_c(t)$ using the basis $\{|J, M_J, \nu\rangle\}$ and yields

$$\rho_c(t) = \left(A^1_{JM_J}(t)\right)^2|1, 1\rangle\langle 1, 1| + \left(B^1_{JM_J}(t)\right)^2|1, -1\rangle\langle 1, -1| + \left(C^1_{JM_J}(t)\right)^2|1, 0\rangle\langle 1, 0|. \tag{29}$$

Thus, considering that

$$|1, 1\rangle = |\uparrow\uparrow\rangle, \tag{30}$$

$$|1, -1\rangle = |\downarrow\downarrow\rangle, \tag{31}$$

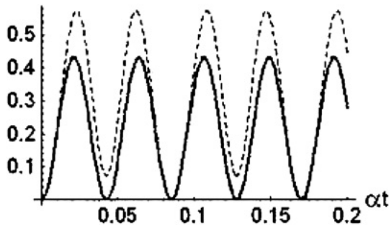


**Figure 1.** $(\Delta S_z)^2$ (dashed line) and $2b(t)$ (bold line) against $\alpha t$ in correspondence to $k = 2$ and $N = 100$.

and

$$|1, 0\rangle = \frac{1}{\sqrt{2}}\{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\}, \tag{32}$$

it is easy to convince oneself that $\rho_c(t)$ assumes the form given by equation (9) with

$$a(t) = \left(A^1_{JM_J}(t)\right)^2, \qquad b(t) = \left(C^1_{JM_J}(t)\right)^2, \qquad e(t) = \left(B^1_{JM_J}(t)\right)^2, \tag{33}$$

when expressed in the standard decoupled basis on $A$ and $B$. It is easy to persuade oneself that also for $M_S = 0$ and $M_S = -1$ the operator $\rho_c(t)$ assumes the form (9) with

$$a(t) = \left(D^0_{JM_J}(t)\right)^2, \qquad b(t) = \left(A^0_{JM_J}(t)\right)^2 + \left(B^0_{JM_J}(t)\right)^2, \qquad e(t) = \left|E^0_{JM_J}(t)\right|^2, \tag{34}$$

if $M_S = 0$ and

$$a(t) = \left(B^{-1}_{JM_J}(t)\right)^2, \qquad b(t) = \left(C^{-1}_{JM_J}(t)\right)^2, \qquad e(t) = \left(A^{-1}_{JM_J}(t)\right)^2, \tag{35}$$

if $M_S = -1$. Suppose to prepare the two spins of the central system and the remaining $M - 2 \equiv N$ spins in the state,

$$|\psi(0)\rangle = |1, 1\rangle|N/2, -N/2 + k, 1\rangle, \tag{36}$$

where $k = 0, 1, \ldots, N$. Thanks to our results expressed by equation (10), we can state with certainty that at a generic time instant $t$ the two central spins are entangled if and only if $(\Delta S_z)^2 < 2b(t)$. In figure 1, we plot $(\Delta S_z)^2$ and $2b(t)$ in correspondence to $N = 100$ and $k = 2$ as a function of $\alpha t$.

Since condition (10) is not verified whatever the time instant $t$ is, we can easily conclude that starting from the initial condition (36) with $N = 100$ and $k = 2$, the central system is unable to develop quantum correlations.

In the light of the result discussed before, the physical reason of this incapacity of the system to develop entanglement stems from the fact that the time evolution from this initial condition never makes significantly different the probabilities of finding the two spins in the state $|\uparrow\uparrow\rangle$ or $|\downarrow\downarrow\rangle$, respectively. It is of relevance the fact that the same conclusion is reached starting from $|\psi(0)\rangle = |1, 1\rangle|N/2, -N/2 + k, 1\rangle$ with $k \leqslant 50$. The behaviour of the system is instead significantly different when $k$ exceeds 50. As an example, we compare in figure 2, $(\Delta S_z)^2$ and $2b(t)$ for $k = 98$ and $N = 100$.

In this case, there exist different time intervals, in which the two spins $A$ and $B$ are entangled because $(\Delta S_z)^2 < 2b(t)$. Thus, the parameter $k$ controls the ability of the system to generate entanglement in the central system. We expect that the amount of entanglement also depends on the choice of $k$ that in turn determines the $S_z$ fluctuations. This indeed is true as witnessed by figures 3 and 4 where the time evolution of the concurrence is reported. We stress that for $k = 100$ the initial condition is an exact stationary state while for $k = 99$ the two

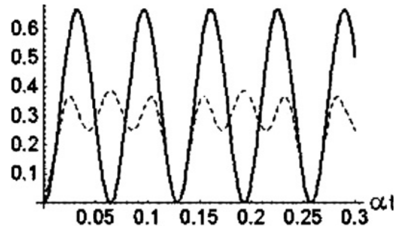


**Figure 2.** $(\Delta S_z)^2$ (dashed line) and $2b(t)$ (bold line) against $\alpha t$ in correspondence to $k = 98$ and $N = 100$.

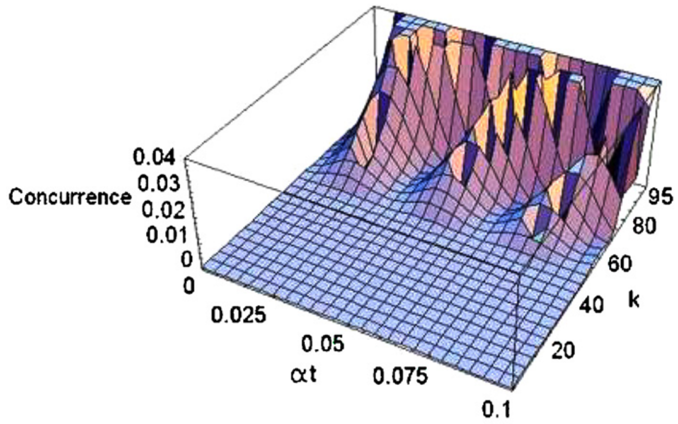


**Figure 3.** Concurrence function of the central system against $\alpha t$ and $k$ in correspondence to the initial condition $|1, 1\rangle|N/2, -N/2 + k\rangle$ for $N = 100$.

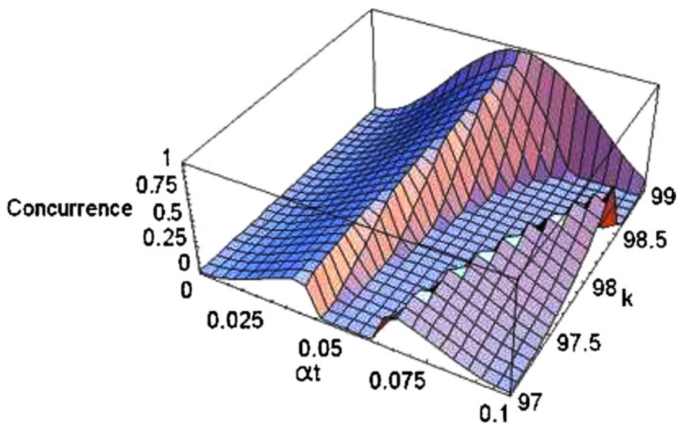


**Figure 4.** Concurrence function of the central system against $\alpha t$ and $k(97 \leqslant k < 100)$ in correspondence to the initial condition $|1, 1\rangle|N/2, -N/2 + k\rangle$ for $N = 100$.

central spins reach the maximum compatible degree of entanglement, since in this case the population of $|{\downarrow\downarrow}\rangle$ exactly vanishes at any time instant so that equation (10) is always fulfilled except when $b(t) = 0$. On this basis, we foresee and we have proved a specular behaviour starting from the initial condition $|1, -1\rangle|N/2, -N/2 + k, 1\rangle$ in the sense that in this case the entanglement reaches its maximum when $k = 1$.

We emphasize that the constraint on the fluctuations of $S_z$ expressed by equation (10) is the key of the simplicity with which we analyse the appearance and the disappearance of entanglement as a function of time and more important to understand its dependence on the value of $k$. In this communication, we have proved that when two two-level systems are described by a density matrix expressed by equation (9) at any time instant $t$, there is a simple and reliable procedure of experimental interest after which the occurrence of quantum correlations be surely claimed or excluded. We propose indeed the measurement of at most two populations which amounts at comparing the variance of $S_z$ with the probability of finding one spin up and the other down. Our approach to the entanglement of the pair of two two-level systems has the merit of directly involving quantities having a clear physical meaning. Applying our criterion to a spin star system, we are able to fully exploit the novelty of our point of view to explain the dependence of the ability of the system to develop entanglement on the initial conditions.

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